

Exact monopole instantons and cosmological solutions in string theory from abelian dimensional reduction ^{*}

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Abstract

We compute the exact string vacuum backgrounds corresponding to the non-compact coset theory $SU(2,1)/SU(2)$. The conformal field theory defined by the level $k = 4$ results in a five dimensional singular solution that factorizes in an asymptotic region as the linear dilaton solution and a S^3 model. It presents two abelian compact isometries that allow to reinterpret it from a four dimensional point of view as a stationary and magnetically charged space-time resembling in some aspects the Kerr-Newman solution of general relativity. The $k = \frac{13}{7}$ theory on the other hand describes a cosmological solution that interpolates between a singular phase at short times and a $S^1 \times S^2$ universe after some planckian times.

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1. Last years have seen a lot of research in string theory addressing the question of interesting vacua, presumably verifying the low energy string equations of Callan et al. [1]. These solutions could tentatively well represent the effective arena in which the string moves, coming from some compactification from 26 or 10 dimensions to the usual 4 [2]. In relation with this mechanism the old Kaluza-Klein idea resorted in the context of string theory at last time [3]. String solutions of this type naturally arise in the form of exactly solvable two-dimensional sigma models, the so called gauged Wess-Zumino-Witten models (GWZWM's) [4], if the gauged group is not a maximal one, but an invariant subgroup of it [5]. In this paper we present a non trivial example of this mechanism based on the $SU(2, 1)/SU(2)$ coset model.

2. It is well known that the Weyl invariance condition of the two dimensional sigma model representing a bosonic string moving on graviton-axion-dilaton d dimensional backgrounds (G, B, D) imposes that at one loop they satisfy the set of equations [1]

$$\begin{aligned} 0 &= R_{ab} - \nabla_a \nabla_b D - \frac{1}{4} H_{acd} H_b{}^{cd} \\ 0 &= \nabla^c (e^D H_{abc}) \\ \Lambda &= \frac{1}{6} H_{abc} H^{abc} - e^{-D} \nabla^a \nabla_a e^D \end{aligned} \quad (1)$$

where $H \equiv dB$ and $\Lambda = \frac{2}{3}(d-26)k$ ($k = \frac{1}{2\alpha'}$ in string notation).

On the other hand GWZWM's are exactly solvable two dimensional conformal models that explicitly realize the G/H coset models of current algebra, and give rise to a sigma model with specific backgrounds defined as follows.¹ If we pick a basis $\{T_a, a = 1, \dots, \dim H\}$ in \mathcal{H} (Lie algebra of H), then by integrating out the gauge fields we obtain the one loop order effective action

$$\begin{aligned} I_{eff}[g] &= \frac{k}{4\pi} (I_{WZ}[g] + \tilde{I}[g]) - \frac{1}{8\pi} \int_{\Sigma} D(g) * R^{(2)} \\ \tilde{I}[g] &= \int_{\Sigma} \frac{1}{l} (\lambda^c)^{ab} \text{tr}(T_a g^{-1} dg) \wedge (* - i1) \text{tr}(T_b dg g^{-1}) \end{aligned} \quad (2)$$

where $g \in G$, I_{WZ} is the WZ action, and $l = l(g)$ and $\lambda^c = \lambda^c(g)$ are the determinant and the cofactor matrix of

$$\lambda_{ab}(g) = \text{tr}(T_a T_b - g T_a g^{-1} T_b) \quad (3)$$

Clearly the gauge invariance condition $I_{eff}[hgh^{-1}] = I_{eff}[g]$, $h \in H$, makes the effective target dependent on $d = \dim G - \dim H$ gauge invariant field variables constructed from g . The d dimensional metric and torsion are then read from (2). The dilaton field appearing in the term linear in the world-sheet curvature $R^{(2)}$ is given by

$$D(g) = \ln |l(g)| + \text{constant} \quad (4)$$

¹For full details and conventions we refer the reader to Section 2 and appendices of [6].

and comes from the determinant in the gaussian integration that leads to (2).

In reference [6] models of this type based on the $SU(2,1)/U(2)$ coset were considered. Here we will consider the gauging of a $SU(2)$ non maximal subgroup and the resulting one loop backgrounds. From general arguments the coset model $SU(2,1)/SU(2)$ will lead to a five dimensional space-time of minkowskian signature. Now let us recall some facts described at length in [6].

An arbitrary element $g \in SU(2,1)$ may be locally parametrized as follows,

$$g = H(N^\dagger, 1) e^{r\lambda_4} e^{i\frac{t}{2}\lambda_8} H(X, 1)H(N, 1) \quad (5)$$

where $H(A, 1)$ is the $SU(2,1)$ embedding of the $SU(2)$ matrix A . This parametrization breaks down at $r = 0$ (where the solution will have a singularity), but it is clear that N will result gauged away in any case. The target manifold that results is then isomorphic to $S^1 \times \mathfrak{R} \times S^3$. We choose for $X \in SU(2)$ the following Euler parametrization,

$$X = e^{i\frac{\psi+\theta}{2}\sigma_3} e^{i\xi\sigma_2} e^{i\frac{\psi-\theta}{2}\sigma_3} \quad (6)$$

Then the remaining five gauge invariant variables that will locally parametrize the effective target are the “radius” $0 < r < \infty$, with $r \rightarrow 0$ the singular region and $r \rightarrow \infty$ the “asymptotic” one, the periodic variables $0 \leq \frac{t}{\sqrt{3}}, \psi, \theta < 2\pi$, and the azimuthal angle $0 \leq \xi \leq \pi/2$.

3. Here we present the one loop backgrounds of our model. The parametrization (5) (with $N = 1$) will be assumed. Let us introduce the following non negative functions,

$$\begin{aligned} h &= h(r, \psi, \xi) \equiv [c^2 + 3 - 2 \cos^2 \xi (1 + c \cos 2\psi)]^{\frac{1}{2}} \\ f &= h(r, \psi, 0) \equiv |c e^{i2\psi} - 1| \\ p &= p(r, \psi, \xi) \equiv [1 + 3 \frac{s^2}{h^2} \sin^2 \xi]^{\frac{1}{2}} \end{aligned} \quad (7)$$

where $c \equiv \cosh r$, $s \equiv \sinh r$. Then the computations before explained being lengthy but straightforward are similar to those in [6] (a convenient basis in $\mathcal{H} = \mathcal{SU}(2)$ is given by the Gell-Mann matrices $\{ \lambda_1, \lambda_2, \lambda_3 \}$) and lead to the following results: if we introduced the basis of one-forms

$$\begin{aligned} \omega^0 &= \frac{p}{2} (dt - 2\sqrt{3} \frac{f^2}{p^2 h^2} \cos^2 \xi \omega_\psi) \\ \omega^1 &= dr \\ \omega^2 &= \frac{s}{f} d\xi \\ \omega^3 &= \frac{2}{p} \frac{c}{s} \frac{f}{h} \cos \xi \omega_\psi \\ \omega^4 &= \frac{s}{h} \sin \xi d\tilde{\theta}, \quad \tilde{\theta} \equiv \theta + \frac{\sqrt{3}}{2} t \end{aligned} \quad (8)$$

where

$$\omega_\psi = d\psi + \frac{2c}{f^2} \sin 2\psi \tan \xi d\xi \quad (9)$$

and its dual “fünfbein” in the tangent space ($e_a(\omega^b) = \delta_a^b$)

$$\begin{aligned} e_0 &= \frac{2}{p} \partial_t \\ e_1 &= \frac{f}{\partial_r} \\ e_2 &= \frac{f}{s} (\partial_\xi - \frac{2c}{f^2} \sin 2\psi \tan \xi \partial_\psi) \\ e_3 &= \frac{p}{2 \cos \xi} \frac{s}{c} \frac{h}{f} (\partial_\psi + \frac{2\sqrt{3}}{p^2} \frac{f^2}{h^2} \cos^2 \xi \partial_t) \\ e_4 &= \frac{h}{s} \csc \xi \partial_{\tilde{\theta}} \end{aligned} \quad (10)$$

then the backgrounds may be expressed as

$$\begin{aligned} G &= \eta_{ab} \omega^a \otimes \omega^b, \quad \eta \equiv \text{diag}(-1, 1, 1, 1, 1) \\ B &= \frac{\sqrt{3}}{2} \frac{s}{c} \frac{h}{\sqrt{c^2 + 3}} \csc \xi \omega^0 \wedge \omega^4 - \frac{f}{\sqrt{c^2 + 3}} \cot \xi \omega^3 \wedge \omega^4 \\ D &= \ln(s^2 h^2) + D_0 \end{aligned} \quad (11)$$

They obey equations (1) with negative cosmological constant $\Lambda = -12$; this implies a one loop value of the level $k^{(1)} = \frac{6}{7}$, very different from the rational values $k_+ = 4, k_- = \frac{13}{7}$ obtained by imposing that the exact central charge of the model

$$c(k) = \frac{8k}{k-3} - \frac{3k}{k-2} = 5 + 6 \frac{3k-5}{(k-2)(k-3)} \quad (12)$$

cancels the ghost contribution $c_{ghost} = -26$. This fact seems common to GWZWM's; as verified previously in some models the one loop results should be taken with extreme care.

The solution on the other hand has minkowskian signature as anticipated and presents a true singularity at $r = 0$ (those at $h = 0$ or $f = 0$ are included there). This can be seen from the computation (details of which we skip) of some scalar invariants; as an example we write down the important ones of dimension two related to B and D respectively

$$\begin{aligned} I_1 &\equiv -\frac{1}{24} H^2 + 1 = -\frac{1}{s^2} + \frac{1}{h^2} (4 - 3 \sin^4 \xi - 4 \cos^2 \xi c \cos 2\psi) \\ &+ 6 \frac{\sin^4 \xi}{h^4} (1 + \sin^2 \xi - \cos^2 \xi c \cos 2\psi) \\ I_2 &\equiv \frac{1}{4} \nabla_a D \nabla^a D - 4 = \frac{1}{s^2} - \frac{1}{h^2} (4 + 6 \sin^2 \xi + 3 \sin^4 \xi - 4 \cos^2 \xi c \cos 2\psi) \\ &+ 6 \frac{\sin^4 \xi}{h^4} (1 + \sin^2 \xi - \cos^2 \xi c \cos 2\psi) \end{aligned} \quad (13)$$

The scalar curvature is given by $R = 6 - 10I_1 - 4I_2$ and can be written as

$$\frac{1}{6} R = 1 + \frac{1}{s^2} + \frac{4}{h^2} (-1 + c \cos 2\psi) \cos^2 \xi + 7 \frac{s^2}{h^4} \sin^4 \xi \quad (14)$$

Furthermore the curvature tensor does not display a flat region. In the large r limit we have ($R \equiv \frac{1}{2}e^r$)

$$\begin{aligned} G &\xrightarrow{r \gg 1} -d\tilde{t}^2 + dr^2 + d^2\Omega_3 = \frac{1}{R^2}(-R^2 d\tilde{t}^2 + dR^2 + R^2 d^2\Omega_3) \\ d^2\Omega_3 &= d\xi^2 + \cos^2 \xi d\tilde{\psi}^2 + \sin^2 \xi d\tilde{\theta}^2 \\ B &\xrightarrow{r \gg 1} -\cos^2 \xi d\tilde{\psi} \wedge d\tilde{\theta} \\ D &\xrightarrow{r \gg 1} 4r + D_0 - \ln 16 \end{aligned} \quad (15)$$

where we have introduced the change of variables (of importance later)

$$\begin{aligned} \tilde{\psi} &= \psi + \frac{\sqrt{3}}{2} t \\ \tilde{t} &= t \end{aligned} \quad (16)$$

and $d^2\Omega_{(3)}$ is the standard metric on S^3 ; in fact by introducing the azimuthal coordinates $0 \leq \theta_i \leq \pi$, $i = 1, 2$, by

$$\begin{aligned} \tan \tilde{\psi} &= \tan \theta_1 \cos \theta_2 \\ \sin \xi &= \sin \theta_1 \sin \theta_2 \end{aligned} \quad (17)$$

$$\begin{aligned} \cos \theta_1 &= \cos \tilde{\psi} \cos \xi \\ \sin \theta_2 &= \frac{\sin \xi}{\sqrt{1 - \cos^2 \tilde{\psi} \cos^2 \xi}} \end{aligned} \quad (18)$$

we get the standard form

$$d^2\Omega_3 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\tilde{\theta}^2 \quad (19)$$

On the other hand the axionic stress field H takes the form

$$H \xrightarrow{r \gg 1} -2 \sin^2 \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\tilde{\theta} = -2 \epsilon^{(3)} \quad (20)$$

where $\epsilon^{(3)}$ is the standard volume form on unit S^3 , $\int_{S^3} \epsilon^{(3)} = 2\pi^2$. With this in mind and decompactifying t (fact anyway irrelevant due to the related isometry) we can see the manifold as $\mathfrak{R}^2 \times S^3$, with the $r = \text{constant}$ submanifolds being topologically $\mathfrak{R} \times S^3$, that at $r = 0$ being singular. From the conformal field theory point of view the asymptotic geometry given by (15) corresponds to no other thing that the product of the linear dilaton vacuum solution and a level k $SU(2)$ WZWM with axionic charge

$$Q_{axion} = -\frac{1}{4\pi^2} \int_{S^3} H \quad (21)$$

depending in principle on the embedding. For the trivial one is straightforward to get $Q_{axion} = 1$; but because $\pi_3(S^3 \times \mathbb{R}^2) = \mathbb{Z}$ we have that it always will be an integer, and then $k \in \mathbb{Z}$ should hold for a consistent quantization of the model. In particular the conformal value $k_+ = 4$ should be a good one, and in fact we will present evidence in Section 5 that the perturbative theory corresponds to this case.

We remark this asymptotic form of the five dimensional solution is that corresponding to a particular limit of the ten dimensional charged black five-brane solutions obtained in the low energy limit of the superstring in reference [11]; our solution is in a wide sense an instanton that interpolates between this phase and the singular one at $r = 0$.

Finally we mention that the “dual” solution (in the sense of reference [7]) to (11) related to the traslational isometry in the $\tilde{\theta}$ direction results torsionless, and is in fact the tensor product of a one dimensional space (a scalar field of “wrong” sign from the world-sheet point of view) and the solution found in [6] for the $SU(2,1)/U(1)$ model; this result could have been conjectured from the algebraic coset equality $\mathcal{G}/\mathcal{H}_1 \equiv \mathcal{G}/(\mathcal{H}_1 \times \mathcal{H}_2) \times \mathcal{H}_2$ and the field theoretic equivalence between both theories; however as we will see the five dimensional solution (11) admits very interesting interpretations via dimensional reduction.

4. Let us remember here some basic facts about abelian Kaluza-Klein dimensional reduction. Let us assume that we have our fields in $d+1$ dimensions and an abelian isometry in the x coordinate direction, i.e. we can introduce a one-form

$$\omega^x = e^\chi (dx + a) \quad (22)$$

in such a way that

$$\begin{aligned} G^{(d+1)} &= G + \eta_{xx} (\omega^x)^2 \\ B^{(d+1)} &= B + e^{-\chi} b \wedge \omega^x \end{aligned} \quad (23)$$

where both gauge fields a, b and G, B are d dimensional in the orthogonal directions to the space ($\eta_{xx} = 1$) or time ($\eta_{xx} = -1$) like direction x . Then by working out the $d+1$ dimensional objects, we get that the equations (1) in terms of the d dimensional fields (G, a, χ, B, b, D) can be derived from the effective action

$$\begin{aligned} S &= \int d^d x \sqrt{|G|} e^{\tilde{D}} (R - \Lambda + \nabla^a \tilde{D} \nabla_a \tilde{D} - \nabla^a \chi \nabla_a \chi - \frac{1}{12} \tilde{H}^2 \\ &- \frac{\eta_{xx}}{4} (e^{2\chi} F[a]^2 + e^{-2\chi} F[b]^2)) \end{aligned} \quad (24)$$

where $F[A] = dA$ is the gauge strength tensor, $\tilde{D} \equiv D + \chi$ and

$$\tilde{H} = H - F[a] \wedge b \quad (25)$$

is a sort of generalization of the well-known Chern-Simons completion [8]. The action (24) contains the bosonic part of $d = 10, N = 1$ SUGRA (fact already noted

in ref. [1] in relation to the original effective string action) coupled to SUSY QED [9], the last coupling being correctly reproduced by the dimensional reduction, and then reproducing the bosonic sector of the low energy heterotic string.² It also translates the original $d + 1$ dimensional reparametrization and axionic invariances into the d dimensional ones plus the standard gauge invariance in a and

$$\begin{aligned} b &\longrightarrow b + d\phi \\ B &\longrightarrow B + \phi F[a] \end{aligned} \quad (26)$$

with arbitrary ϕ , which leaves (25) invariant.

Going to our model, having two abelian isometries there are two possible dimensional reductions to consider to get $d = 4$ interpretations of our $d = 5$ solution. The dilaton field will always be given as in (11).

The first possibility, the “instantonic” one, is related to the t translational isometry and in some sense is a particular one, because in general we should not expect isometries for a general GWZWM with maximal gauge group but it is present in the model of reference [6] and remains here; however e.g. the models of [10] do not present Killing symmetries at all. By identifying $x \equiv t$ ($\eta_{tt} = -1$), we get from (11) and (23) the following backgrounds

$$\begin{aligned} G &= \delta_{ab} \omega^a \otimes \omega^b, \quad a, b = 1, 2, 3, 4 \\ B &= -\frac{1}{2} \frac{f}{p} \frac{1}{c} \cot \xi \omega^3 \wedge \omega^4 \\ \chi &= \ln \frac{p}{2} \\ a &= -2\sqrt{3} \frac{f^2}{p^2 h^2} \cos^2 \xi \omega_\psi \\ b &= -\frac{3}{2} \frac{s}{h} \sin \xi \omega^4 \end{aligned} \quad (27)$$

It can be interpreted as some kind of “dyonic”, static instanton, with a and b giving rise, as seen by the e_4 observers wrt the Wick rotated and decompactified “time” variable $\tau = -i\tilde{\theta}$, respectively to a magnetic field

$$\begin{aligned} B_{inst} &= \sum_{a=1}^3 B_a \omega^a \\ B_1 &= \frac{4\sqrt{3} \sin \xi}{f^2 p h} \left(\frac{2c}{p^2 h^2} (f^4 + 4c^2 \sin^2 2\psi \sin^2 \xi) + (c^2 + 1) \cos 2\psi - 2c \right) \\ B_2 &= \frac{8\sqrt{3} s^2}{f p^3 h^3} (c \cos 2\psi - 1) \sin^2 \xi \cos \xi \\ B_3 &= \frac{2\sqrt{3} s^2}{f p^2 h^2} \sin 2\psi \sin 2\xi \end{aligned} \quad (28)$$

² More precisely, it is so for the solutions with $\eta_{xx} = 1$, $b = a$, $\chi = 0$ and $\Lambda = 0$; we remark our current solution is not supersymmetric.

and to an electrostatic field

$$E_{inst} = -dV \quad , \quad V = 3 \frac{s}{h} \sin \xi \quad (29)$$

Its action is fixed by the dilaton value at infinity or, equivalently, by the string coupling constant measured by asymptotic observers (r_c is a cutoff in the r -integration)

$$S_{inst.} = 2\sqrt{3} \pi^3 e^{D_0 - \ln 16 + 4r_c} = \frac{2\sqrt{3} \pi^3}{g_{string}^2} \quad (30)$$

We can also tentatively adscribe a magnetic charge by

$$Q_{magn} \equiv \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int_{S^2} d\vec{\Sigma} \cdot \vec{B} = \sqrt{3} \quad (31)$$

An analogous definition leads to a null value for the electric charge; the asymptotic expansion for the potential

$$V = 3\sqrt{\frac{4\pi}{3}} Y_1^0(\frac{\pi}{2} - \xi, \psi) + 6\sqrt{\frac{4\pi}{105}} \left(Y_3^2(\frac{\pi}{2} - \xi, \psi) + Y_3^{-2}(\frac{\pi}{2} - \xi, \psi) \right) \frac{1}{c} + O(\frac{1}{c^2}) \quad (32)$$

could lead to assign multipolar moments of order 1 and 3 (and higher) to the field; however the facts that the asymptotic metric is not flat (neither the S^2 metric is the standard one!), certainly $\nabla^2 V \neq 0$, and τ does not correspond to the proper time measured by some “privileged” asymptotic e_4 observers (only those near $\xi = \frac{\pi}{2}$ measure this time) obscure this interpretation.

A much more appealing space-time of minkowskian signature $\eta = \text{diag}(-1, 1, 1, 1)$ is obtained by considering the isometry related to $x \equiv \theta$. It is a “natural” one because it is originated in the non-maximal gauging. Furthermore, it is compact and space like ($\eta_{\theta\theta} = 1$) as in the original spirit of Kaluza, and present no torsion and other scalar field besides the dilaton. The non-zero four dimensional fields are in this case

$$\begin{aligned} G &= \eta_{ab} \omega^a \otimes \omega^b, \quad a, b = 0, 1, 2, 3 \\ \chi &= \ln\left(\frac{s}{h} \sin \xi\right) \\ b &= \frac{1}{p} \frac{s}{h} \left(\sqrt{3} \frac{s}{h} \sin^2 \xi \omega^0 - \frac{f}{2c} \cos \xi \omega^3 \right) \end{aligned} \quad (33)$$

For large r we have (see (15))

$$\begin{aligned} G &\xrightarrow{r \gg 1} -dt^2 + dr^2 + d\xi^2 + \cos^2 \xi d\tilde{\psi}^2 \\ \chi &\xrightarrow{r \gg 1} \ln \sin \xi \\ b &\xrightarrow{r \gg 1} -\cos^2 \xi d\tilde{\psi} \\ D &\xrightarrow{r \gg 1} 4r + D_0 - \ln 16 \end{aligned} \quad (34)$$

that is the wormhole “throat” solution, a limiting case of the extremal member of the family of solutions to the heterotic string found in [14, 15].

Let us closely analyze this solution. To this end it seems instructive to us to compare it with the Kerr-Newman solution (KNS) of General Relativity (GR); both are related in many aspects (but being very different physically!) as we will see.

In the study of stationary space-times in GR is usual to define two classes of observers in the following way (see e.g. [16]). The stationary observers (SO) are those who follow the orbits of the time-like Killing vector. In appropriate coordinates where the time variable “ t ” is taken as the parameter of the flux lines they have constant space coordinates, and their space-time measurements result time independent. In our solution the field e_0 corresponds to the (normalized) Killing field; the basis used in (33) is then a good one for these observers. They have orbits given by

$$(r(t), \xi(t), \psi(t)) = (r_0, \xi_0, \psi_0) \quad (35)$$

measure proper time

$$\tau^{(SO)} = \frac{1}{2} p(r_0, \xi_0, \psi_0) (t - t_0) \quad (36)$$

and its spatial metric is

$$H^{(SO)} = \delta_{ij} \omega^i \otimes \omega^j \quad (37)$$

On the other hand if a function “time” t defining simultaneity space-like surfaces Σ_t of $t = \text{constant}$ is given, it is usual to refer the measurements to “fiducial observers” (FO) defined to be those whose world lines are orthogonal to them. Let us then introduce the vector fields

$$\begin{aligned} e_0^{(FO)} &= \frac{\sqrt{c^2 + 3}}{c} \left(\partial_t - \frac{\sqrt{3}}{2} \frac{s^2}{c^2 + 3} \partial_\psi \right) \\ e_3^{(FO)} &= \frac{s}{\sqrt{c^2 + 3}} \frac{h}{f} \sec \xi \partial_\psi \end{aligned} \quad (38)$$

and their dual one-forms

$$\begin{aligned} \omega_{(FO)}^0 &= \frac{c}{\sqrt{c^2 + 3}} dt \\ \omega_{(FO)}^3 &= \frac{f}{h} \cos \xi \left(\frac{\sqrt{c^2 + 3}}{s} \omega_\psi + \frac{\sqrt{3}}{2} \frac{s}{\sqrt{c^2 + 3}} dt \right) \end{aligned} \quad (39)$$

Their are related to (e_0, e_3) and (ω^0, ω^3) respectively by a two dimensional local Lorentz transformation of parameter $\beta^{(FO)}$ given by

$$\tanh \beta^{(FO)} = \frac{\sqrt{3}}{2} \frac{f}{h} \frac{s}{c} \cos \xi \quad (40)$$

Then the orbits of our FO wrt t coordinate are precisely those of the $e_0^{(FO)}$ vector field,

$$\begin{aligned}
r(t) &= r_0 \\
\xi(t) &= \xi_0 \\
\psi(t) &= \psi_0 + \Omega^{(FO)}(r_0) (t - t_0) \quad , \quad \Omega^{(FO)}(r) = -\frac{\sqrt{3}}{2} \frac{s^2}{c^2 + 3}
\end{aligned} \tag{41}$$

and the proper time measured by them is

$$\tau^{(FO)} = \frac{c_0}{\sqrt{c_0^2 + 3}} (t - t_0) \tag{42}$$

It is clear from here that they rotate with constant coordinate angular velocity $\Omega^{(FO)} = \frac{d\psi(t)}{dt}$ ($2\sqrt{3}(c^2 + 3)^{-1}$ relative to distant FO), and have zero angular momentum $J = e_0^{(FO)} \cdot \partial_\psi$ wrt the *asymptotic* Killing field ∂_ψ (they are the “locally non rotating observers” of [17]). However these FO do not seem “natural” in the sense that the absolute value of their angular velocity is null in the singular region ($\Omega^{(FO)} \xrightarrow{r \rightarrow 0} -\frac{\sqrt{3}}{8} r^2$) and grows when we approach the asymptotic region until it reaches the value $\Omega_\infty^{(FO)} = -\frac{\sqrt{3}}{2}$; we remember that in the KNS they have zero asymptotic angular velocity and in fact they coincide there with the SO. A hint to search for more natural observers is obtained by looking at the spatial metric of the FO

$$\begin{aligned}
H^{(FO)} &\equiv G + \omega_{(FO)}^0 \otimes \omega_{(FO)}^0 = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega_{(FO)}^3 \otimes \omega_{(FO)}^3 \\
&\xrightarrow{r \gg 1} dr^2 + d\xi^2 + \cos^2 \xi d\tilde{\psi}^2
\end{aligned} \tag{43}$$

where $\tilde{\psi}$ was introduced in (16). From here we see that the right asymptotic angular coordinate is $\tilde{\psi}$ and not ψ , and then we are tempted to identify special observers that we denote “BH”, analogous to the flat observers of Kerr-Newman space-time, static over the sphere S^2 parametrized by $(\tilde{\psi}, \xi)$ at radius r . We are then lead to introduce a third basis

$$\begin{aligned}
e_0^{(BH)} &= \frac{1}{F_0} \left(\partial_t - \frac{\sqrt{3}}{2} \partial_\psi \right) = \frac{1}{F_0} \partial_{\tilde{t}} \\
e_3^{(BH)} &= \frac{1}{F_0} \left(\frac{s h}{c f \cos \xi} (1 - 3 \frac{\sin^2 \xi}{h^2}) \partial_\psi - \frac{2\sqrt{3} f}{s c h} \cos \xi \partial_t \right) \\
&= F_0 \frac{s}{c} \frac{h}{f} \sec \xi \left(\partial_{\tilde{\psi}} - \omega \partial_{\tilde{t}} \right)
\end{aligned} \tag{44}$$

togheter with their corresponding dual one-forms

$$\begin{aligned}
\omega_{(BH)}^0 &= F_0 (d\tilde{t} + \omega \omega_{\tilde{\psi}}) \\
\omega_{(BH)}^3 &= \frac{1}{F_0} \frac{f c}{h s} \cos \xi \omega_{\tilde{\psi}}
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
\omega_{\tilde{\psi}} &\equiv \omega_{\psi} + \frac{\sqrt{3}}{2} dt = d\tilde{\psi} + \frac{2c}{f^2} \tan \xi \sin(2\tilde{\psi} - \sqrt{3}\tilde{t}) d\xi \\
F_0^2 &= \frac{1}{s^2} \left(c^2 - 4 + 12 \frac{\sin^2 \xi}{h^2} \right) \\
\omega &= \frac{2\sqrt{3}}{F_0^2} \frac{f^2}{s^2 h^2} \cos^2 \xi
\end{aligned} \tag{46}$$

that are also related to (e_0, e_3) and (ω^0, ω^3) by a local Lorentz transformation of parameter $\beta^{(BH)}$ that for sake of completeness we quote

$$\tanh \beta^{(BH)} = \frac{\sqrt{3} f c}{2 h s} \frac{\cos \xi}{1 - 3 \frac{\sin^2 \xi}{h^2}} \tag{47}$$

The spatial metric associated

$$H^{(BH)} = dr^2 + \frac{s^2}{f^2} d\xi^2 + \frac{1}{F_0^2} \frac{c^2}{s^2} \frac{f^2}{h^2} \cos^2 \xi (\omega_{\tilde{\psi}})^2 \tag{48}$$

is that used by these observers moving through the flux lines of vector field $e_0^{(BH)}$

$$(r(\tilde{t}), \xi(\tilde{t}), \tilde{\psi}(\tilde{t})) = (r_0, \xi_0, \tilde{\psi}_0) \tag{49}$$

However, because of the “wave front” dependence of $H^{(BH)}$ (see (44-46)), away the asymptotic region the BH observers do not see the space-time as stationary. In particular the electromagnetic fields they measure are \tilde{t} -dependent; the definition (31) gives null electric charge and $Q_{magn} = \frac{1}{2}$ for this solution. Furthermore, they do not exist beyond the surface $F_0^2 = 0$; inside this surface $e_0^{(BH)}$ becomes space-like. On this surface $||\partial_{\tilde{t}}|| = -F_0^2$ is null, being very reminiscent of the ergosphere in KNS; however this surface is not null and on it their velocity become singular. Also it was not possible for us to think it as an horizon hiding the singularity at $r = 0$, and then a black hole interpretation of the solution is not clear to us.

A digression on the mass to be assigned to the solution is in order. We feel a good definition in a stationary space-time is Komar’s one that we shortly sketch (see e.g. [17]): let a space-time be with an asymptotic spatial region characterized by some $r \rightarrow \infty$ limit, topologically S^2 and $\xi = \xi^a e_a$ an asymptotic time-like Killing vector field; then

$$M = -\frac{1}{8\pi} \int_{S^2} *d\omega^{(\xi)} \tag{50}$$

where $\omega^{(\xi)} = \xi_a \omega^a$ is the dual one-form of ξ . This definition stems from the fact that

$$d * d\omega^{(\xi)} = 2 \xi^b R_{ab} * \omega^a \tag{51}$$

is zero in a flat region, what allows to make (50) “radius” - independent so that to have a sensible definition for the isolated system. In any case it is possible to take

the limit $r \rightarrow \infty$ on the RHS of (50) if (51) goes asymptotically to zero, and in fact it is made for example in the computation relative to KNS.

If we applied this definition to (33), we obtain

$$M = 6 e^{-2r_c} \xrightarrow{r_c \rightarrow \infty} 0 \quad (52)$$

However, in GR we have the gravity equation

$$E_{ab} \equiv R_{ab} - \frac{1}{2} G_{ab} R = 8 \pi T_{ab} \quad (53)$$

and because of

$$\nabla^a (E_{ab} \xi^b) = \xi^b \nabla^a E_{ab} + E_{ab} \nabla^a \xi^b = 0 \quad (54)$$

for any Killing field ξ , the form $E_{ab} \xi^b \omega^a$ is conserved and allows to define a charge on a space-like $(d-1)$ -dimensional volumen V as

$$Q^\xi = C \int_V E_{ab} \xi^b * \omega^a \quad (55)$$

where C is a constant. In our context equation (53) naturally arises if we introduce the Einstein metric

$$G^E \equiv e^{\tilde{D}} G \quad (56)$$

The backgrounds $(G^E, b, \tilde{D}, \chi)$ then result classical solutions of the Einstein action

$$\begin{aligned} S_E[G^E, b, \tilde{D}, \chi] &= \int_{\mathcal{M}} \left(*R^E - \frac{1}{2} \nabla^E \tilde{D} \wedge * \nabla^E \tilde{D} - \nabla^E \chi \wedge * \nabla^E \chi \right. \\ &\quad \left. - \frac{1}{4} e^{-\tilde{D}-2\chi} F_E[b] \wedge * F_E[b] - * \Lambda e^{-\tilde{D}} \right) \end{aligned} \quad (57)$$

It is possible to show that both definitions (50) and (55) with ξ the time-like Killing field ($C = \frac{1}{8\pi}$ in this case) applied to G^E coincides, and yield

$$M = \frac{1}{32} g_{string}^{-2} \quad (58)$$

We see that the mass ³ (as the instantonic action (30)) is determined by the dilaton value in the asymptotic region.

³ In string units, $M = g_{string}^{-2} \frac{1}{\sqrt{8\alpha'}}$.

5. Here we present the exact solution for the metric and dilaton fields, computed by using an ansatz guessed from algebraic and gauge invariance arguments in References [10]. To this end we first introduce some notation. We will refer the indices to the generators given by $\{\lambda_i, i = 1, 2, 3; \lambda_8; \lambda_1^\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5); \lambda_2^\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)\}$, where $\{\lambda_1, \dots, \lambda_8\}$ are the Gell-Mann matrices.

If $X = x_0 1 + i \vec{x} \cdot \vec{\sigma}$ is an arbitrary $SU(2)$ element (σ_i are the Pauli matrices, $x_0^2 = 1 - \vec{x}^2$) the adjoint representation is given by the 3×3 matrix

$$R(X)_{ij} \equiv \frac{1}{2} \text{tr}(\sigma_i X \sigma_j X^\dagger) = (2x_0^2 - 1) \delta_{ij} + 2 x_i x_j + 2 x_0 \epsilon_{ijk} x_k \quad (59)$$

and the left and right $SU(2)$ operators are ⁴

$$\hat{\xi}_i^L = x_0 \partial_i - \epsilon_{ijk} x_j \partial_k = -\hat{\xi}_i^R|_{-\vec{x}} \quad (60)$$

We then define the left currents as linear operators on the group manifold G by

$$\hat{L}_a g = -\lambda_a g, \quad g \in G \quad (61)$$

In the parametrization (5) the computations yield

$$\begin{aligned} \hat{L}_i &= i R(N)_{ji} \hat{\xi}_j^R|_X - i \hat{\xi}_i^R|_N \\ \hat{L}_8 &= i 2 \partial_t + i \sqrt{3} (\hat{\xi}_3^R|_X - \hat{\xi}_3^L|_X - \hat{\xi}_3^L|_N) \\ \hat{L}_\alpha^+ &= -\frac{1}{2} N_{1\alpha} (\partial_r - i \sqrt{3} \frac{s}{c} \partial_t) + \vec{A}_\alpha^+ \cdot \vec{\xi}^R|_X + \vec{B}_\alpha^+ \cdot \vec{\xi}^L|_N = (\hat{L}_\alpha^-)^* \end{aligned} \quad (62)$$

where $((\check{e}_i)_j = \delta_{ij})$

$$\begin{aligned} \vec{A}_\alpha^+ &= \frac{-i}{2sc} \left(N_{1\alpha} \left((1 + \frac{3}{2}s^2) R(X)^t - (2c^2 - 1)1 \right) \check{e}_3 + c N_{2\alpha} (R(X)^t - c 1)(\check{e}_1 - i \check{e}_2) \right) \\ \vec{B}_\alpha^+ &= \frac{-i}{2sc} \left(N_{1\alpha} (2c^2 - 1) \check{e}_3 + c^2 N_{2\alpha} (\check{e}_1 - i \check{e}_2) \right) \end{aligned} \quad (63)$$

Similarly we define the right currents by

$$\hat{R}_a g = g \lambda_a, \quad g \in G \quad (64)$$

and compute them to get ($u \equiv e^{i\frac{\sqrt{3}}{2}t}$)

$$\begin{aligned} \hat{R}_i &= -i R(N)_{ji} \hat{\xi}_j^R|_X \\ \hat{R}_8 &= -i 2 \partial_t \\ \hat{R}_\alpha^+ &= \frac{u}{2} (XN)_{1\alpha} (\partial_r + i \sqrt{3} \frac{s}{c} \partial_t) + \vec{A}_\alpha^+ \cdot \vec{\xi}^L|_X + \vec{B}_\alpha^+ \cdot \vec{\xi}^L|_N = (\hat{R}_\alpha^-)^* \end{aligned} \quad (65)$$

⁴ In fact they obey

$$\hat{\xi}_i^L(X) = i \sigma_i X, \quad \hat{\xi}_i^R(X) = i X \sigma_i$$

When necessary we will indicate explicitly the $SU(2)$ element we are referring to.

where

$$\begin{aligned}\vec{A}_\alpha^+ &= \frac{i u}{2 s c} \left((X N)_{1\alpha} \left(\left(1 + \frac{s^2}{2} \right) 1 - R(X) \right) \check{e}_3 + c (X N)_{2\alpha} (c 1 - R(X)) (\check{e}_1 - i \check{e}_2) \right) \\ \vec{B}_\alpha^+ &= \frac{i u}{2 s c} ((X N)_{1\alpha} \check{e}_3 + c (X N)_{2\alpha} (\check{e}_1 - i \check{e}_2))\end{aligned}\quad (66)$$

By construction both set of currents satisfy the corresponding λ_a -algebra. Now we introduce the Casimir operators ($g_{ab} = \text{tr} T_a T_b$)

$$\Delta_G^L = g^{ab} \hat{L}_a \hat{L}_b \quad (67)$$

and the Virasoro-Sugawara laplacian associated with the coset $G/H = SU(2, 1)/U(1)$

$$\hat{L}_0^L = \frac{1}{k-3} \Delta_G^L - \frac{1}{k-2} \Delta_H^L \quad (68)$$

Analog construction in the right sector.

Finally we consider gauge invariant functions, i.e.

$$(\hat{L}_i + \hat{R}_i) f(g) = 0 \quad , i = 1, 2, 3 \quad , \quad g \in SU(2, 1) \quad (69)$$

and on this subspace we define the metric and dilaton fields to be those that obey the “hamiltonian” equation ⁵

$$\begin{aligned}\hat{H} f(g) &\equiv \frac{1}{k-3} \chi^{-1} \partial_\mu (\chi G^{\mu\nu} \partial_\nu) f(g) \\ \hat{H} &\equiv \hat{L}_0^L + \hat{L}_0^R = \frac{1}{k-3} \left(\hat{L}_8^2 + 2 \{ \hat{L}_\alpha^+, \hat{L}_\alpha^- \} + \lambda \vec{\hat{L}}^2 \right) \\ \chi &\equiv e^D |\det G|^{\frac{1}{2}}\end{aligned}\quad (70)$$

where

$$\lambda = \frac{1}{k-2} \quad (71)$$

By carrying out the computations we read from these equations the exact backgrounds; let us introduce the functions

$$\begin{aligned}\alpha &= \frac{\sec^2 \xi}{\frac{f^2}{s^2} + \lambda} \left(\frac{h^2 p^2}{4c^2} + \lambda \left(1 + \frac{h^2}{s^2} - \cos^2 \xi \frac{3c^2 + 1}{4c^2} \right) + \lambda^2 \right) \\ -\delta &= \frac{\sec^2 \xi}{c^2} \left(h^2 + \lambda \left((c^2 + 3) \left(1 + \frac{h^2}{s^2} \right) - 4 \cos^2 \xi \right) + \lambda^2 (c^2 + 3) \right) \\ F^2 &= -\delta^{-1} \sec^2 \xi \left(\frac{h^2}{c^2} F_0^2 + \lambda \left(1 + \frac{h^2}{s^2} - \frac{4}{c^2} \cos^2 \xi \right) + \lambda^2 \right) \\ \tilde{\omega} &= \frac{\sqrt{3}}{2} \frac{\frac{f^2}{s^2} + \lambda}{\delta F^2} \left(\frac{s^2}{c^2} + \frac{3s^4}{4\alpha c^4} + \frac{\delta}{\alpha (\frac{f^2}{s^2} + \lambda)} \right)\end{aligned}\quad (72)$$

⁵ The computations in the left and right sectors lead to the same result.

Then the modified “BH” vielbein basis reads

$$\begin{aligned}
\tilde{e}_0^{(BH)} &= F^{-1} \partial_{\tilde{t}} \\
\tilde{e}_1 &= \partial_r \\
\tilde{e}_2 &= \left(\frac{f^2}{s^2} + \lambda \right)^{\frac{1}{2}} \left(\partial_\xi - \frac{2c}{f^2 + \lambda s^2} \sin 2\psi \tan \xi \partial_{\tilde{\psi}} \right) \\
\tilde{e}_3^{(BH)} &= \left(\frac{-\delta F^2}{\frac{f^2}{s^2} + \lambda} \right)^{\frac{1}{2}} (\partial_{\tilde{\psi}} - \tilde{\omega} \partial_{\tilde{t}}) \\
\tilde{e}_4 &= \csc \xi \left(\frac{h^2}{s^2} + \lambda \right)^{\frac{1}{2}} \partial_{\tilde{\theta}}
\end{aligned} \tag{73}$$

from which the metric can be straightforwardly read, and the dilaton field is

$$e^{D-D_0} = s^3 c \cos \xi \left| \delta \left(\frac{h^2}{s^2} + \lambda \right) \right|^{\frac{1}{2}} \tag{74}$$

The reader could ask at this point why we first compute the one loop result instead of giving directly the exact backgrounds; the answer is that, to our knowledge, the conjecture expressed by equations (70) has not been proved (to do this it would probably be needed the knowledge of the exact classical equations). Our results however give more support to them, in particular to the no renormalization of the χ -field.

On the other hand, the asymptotic forms of the line element

$$ds^2 = (k-3) G \tag{75}$$

and the dilaton field are given by

$$\begin{aligned}
ds^2 &\xrightarrow{r \gg 1} -(k-3) d\tilde{t}^2 + (k-3) dr^2 + \frac{k-3}{1+\lambda} d^2\Omega_3 \\
D &\xrightarrow{r \gg 1} 4r + D_0 + \ln \frac{|1+\lambda|^{\frac{3}{2}}}{16}
\end{aligned} \tag{76}$$

The requirement of having a positive central charge (given by equation (12)) for the coset theory under study leads to considering two possible regions for k .⁶ The first one corresponds to $k > 3$, and contains in particular the conformal value $k_+ = 4$; the signature remains $(-+++)$, and the solution presents essentially the same features as the k large limit discussed before, up to some obvious field renormalizations in (\tilde{t}, r) and an asymptotic radius

$$R = \left| \frac{k-3}{1+\lambda} \right|^{\frac{1}{2}} \tag{77}$$

⁶ There exists a third region defined by $k < 0$, but in this case the existence of the perturbative path integral is at least dubious [18], and we will not consider it here.

The second region is defined by $\frac{7}{5} < k < 2$ and the conformal value $k_- = \frac{13}{7}$ that belongs to it probably defines a non perturbative phase of the theory. Because of $\lambda < -\frac{5}{3}$, the signature now becomes $(+ - + +)$; the $r(\tilde{t})$ -coordinate is time (space)-like. In particular the $\tilde{\theta}$ -dimensional reduced solution is naturally interpreted as a cosmological geometry that interpolates a singular universe at the beginning of the times ($r = 0$) with a constant curvature one on $S^1 \times S^2$ of radius $r_0 = \sqrt{3 - k}$ and R respectively after time long enough ($r \rightarrow \infty$).

6. We end with some remarks we believe important.

First of all, the exact solutions as well as the one-loop geometry are not asymptotically flat but asymptotic to constant curvature solutions, they present an “infinite throat”. This seems a persistent feature of string solutions obtained as backgrounds of WZW like-models [12,13,14,6]. However as conjectured in [12] there should exist some singular marginal operator that deforms them to an asymptotically flat region; in any case to our knowledge an exact conformal field theory which interpolates, i.e. in $k > 3$ case, between the throat solution and the asymptotically flat one is not known. And linked with the absence of a flat region (or the knowledge of the interpolating solutions), the definition of the mass of the solution as well as the definition of any conserved charge, remains unclear.

As showed in [10,19], $N = 1$ superconformal extensions does not solve the problem; in particular the backgrounds for type II superstrings (up to a trivial rescaling) are the semiclassical ones studied in section 4.

Some $N = 2$ supersymmetric version of these models, other being more appealing from a phenomenological point of view, might cure this “bad” behavior, at least for space-time supersymmetric solutions which have necessarily $\Lambda = 0$. But here care is needed, this condition is probably necessary but it does not assure at all the flatness. The Gepner projection (guilty of the space-time supersymmetry) for an arbitrary $N=2$ model does not hold and how to implement it at the level of backgrounds is not clear to us. Realizations of some models of this kind were recently pursued in [20]; however they are based in hamiltonian reducing a WZW model defined on a superLie algebra G through the gauging of *nilpotent* subalgebras (for exactness, embeddings of $sl(2|1)$ on G specified by some grading) to obtain *non-critical* string theories; in our context we consider critical strings from the start (i.e., without gravity or Liouville mode, and then with matter central charge 26, $10 > 1$) and gauge a *simple* subalgebra where the integrating out of the gauge fields is well defined; for nilpotent subalgebras in the kind of models considered there it defines only the constraint to carry out the hamiltonian reduction and the way the backgrounds could be obtained and then the space-time interpretation remains obscure to us. Maybe the recent formulation of $(0, 2)$ heterotic like WZW theories [21] could lead to formulate models asymptotically product of four dimensional flat Minkowski and a Kazama-Suzuki type theory. If space-time $N = 1$ supersymmetry holds (think not assured as remarked above) the internal asymptotic space should be of the Calabi-Yau type. Work in these directions as well as non abelian examples of

the Kaluza-Klein mechanism presented here are in progress [22].

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